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A Conceptual Summary of the Application of Free Modules in the Applied Mathematical Science

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Abstract

Modules are one of the topics studied in algebra, furthermore there are free modules which are a subtopic of the module itself. This literature aims to find out the application of free modules in various branches of mathematics, both explicitly described and implicitly discussed in previous literature. So that, this research used the Systematic Literature review to find the applications of module theory, especially the application of free modules in previous research. There are applications of free modules in various fields that are quite often discussed in system control theory, representation theory, cryptography, and mathematical physics. This research reveals that free modules not only play a role in pure algebra theory, but also have practical applications in various other sectors of science. This research is useful for future research to find out the application of modules, especially free modules, to research them in more depth.

Keywords: Free Modules; Control Theory; Cryptography; Coding Theory, Mathematical Physics

I. Introduction

In Algebra, one of the well-known topics is Modules. Familiarly, modules are an extension of vector spaces. So, we will see the relationship between vector spaces, modules, and their applications in applied science, especially the applications of free modules in applied mathematics. The study of the application of free modules is important to discuss because the basic concept has the potential for extensive application studies in various fields of applied mathematics.

Nowadays, studies in modern fields such as cryptography, optimal control theory, and mathematical physics are widely known. These studies have shown a relationship with free modules, whether explicitly or not. Therefore, this study was prepared with the aim of providing a brief understanding of the application of free modules in the field of applied mathematics.

Before discussing the relationship further, let's first look at each definition one by one to recall as a foundation for the next discussion. First, we have to recall about the vector spaces, and then we will go into the modules, specifically the free modules. This is the formal definition of the vector space.

Definition 1 [1] Vector Spaces
Let Γ be a non-empty set with two defined operations: addition and multiplication by a number known as a scalar. With the addition defined for every $\gamma_1, \gamma_2 \in \Gamma$ then $\gamma_1 + \gamma_2$ is the sum of γ_1 and γ_2.

Meanwhile, the scalar multiplication of γ_1 by ρ, denoted $\rho\gamma_1$, is defined for every ρ scalar and $\gamma_1 \in \Gamma$. If the following axioms are satisfied by every $\gamma_1, \gamma_2 \in \Gamma$ and every scalar ρ_1 and ρ_2, then Γ and every scalar ρ_1 and ρ_2, then Γ can be called as a vector space and all members in Γ being called vectors.

1) If $\gamma_1, \gamma_2 \in \Gamma$, then $\gamma_1 + \gamma_2 \in \Gamma$

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2) $\gamma_1 + \gamma_2 = \gamma_2 + \gamma_1$
3) $\gamma_1 + (\gamma_2 + \gamma_3) = (\gamma_1 + \gamma_2) + \gamma_3$
4) There is an element $\eta \in \Gamma$ that is called as a zero vector Γ, then it holds that $\eta + \gamma_1 = \gamma_1 + \eta = \gamma_1$ for all $\gamma_1 \in \Gamma$
5) For all $\gamma_1 \in \Gamma$, there is an element $-\gamma_1 \in \Gamma$ that called as a "negative of γ_1" then it holds $\gamma_1 + (-\gamma_1) = (-\gamma_1) + \gamma_1 = \eta$
6) If b is an arbitrary scalar and $\gamma_1 \in \Gamma$ is arbitrary, then it holds that $\rho\gamma_1 \in \Gamma$
7) $\rho(\gamma_1 + \gamma_2) = \rho\gamma_1 + \rho\gamma_2$
8) $(\rho_1 + \rho_2)\gamma_1 = \rho_1\gamma_1 + \rho_2\gamma_1$
9) $\rho_1(\rho_2\gamma_1) = (\rho_1\rho_2)(\gamma_1)$
10) $1\gamma_1 = \gamma_1$

Modules appear as a generalization of vector spaces when the scalar structure of the field is expanded to a ring [2]. We have recalled the definition of the vector spaces, but we also need to know about the rings. So, before we go into the modules, we have to recall about the rings. Here is the formal definition of the rings.

Definition 2 [3] Rings
A ring $(\Upsilon, +, \cdot)$ is a set Υ with the binary operations $+: \Upsilon \times \Upsilon \to \Upsilon$ (addition) and $\cdot: \Upsilon \times \Upsilon \to \Upsilon$ (multiplication) satisfying the following properties:
(a) $(\Upsilon, +)$ is an abelian group. With an identity element written as 0.
(b) $\upsilon_1 \cdot (\upsilon_2 \cdot \upsilon_3) = (\upsilon_1 \cdot \upsilon_2) \cdot \upsilon_3$ ($\cdot$ is associative).
(c) $\upsilon_1 \cdot (\upsilon_2 + \upsilon_3) = \upsilon_1 \cdot \upsilon_2 + \upsilon_1 \cdot \upsilon_3$ and $(\upsilon_2 + \upsilon_3) \cdot \upsilon_1 = \upsilon_2 \cdot \upsilon_1 + \upsilon_3 \cdot \upsilon_1$ ($\cdot$ is left and right distributive over the $+$).

With the definitions given above, we already have the provisions to enter into the discussion regarding the modules, specifically free modules. We will review it in order, we will go into the modules which are then specifically the free modules. Here is the formal definition of the modules.

Definition 3 [4] Modules
Suppose Υ is an arbitrary ring that has an identity but it is not necessary commutative.
1) The left module over Υ (or it can be written left Υ-module) is an abelian group Ω with a scalar multiplication mapping
$$\circ : \Upsilon \times \Omega \to \Omega$$
that satisfies the following axioms (it will be written $\circ(\upsilon, \omega_1) \mapsto \upsilon\omega_1$ for scalar multiplication, where υ is an element of Υ and ω_1 is an element of Ω). In these axioms, υ_1, υ_2 are arbitrary elements of Υ and ω_1, ω_2 are arbitrary members of Ω.
   i. $\upsilon(\omega_1 + \omega_2) = \upsilon\omega_1 + \upsilon\omega_2$
   ii. $(\upsilon_1 + \upsilon_2)\omega_1 = \upsilon_1\omega_1 + \upsilon_2\omega_1$
   iii. $(\upsilon_1\upsilon_2)\omega_1 = \upsilon_1(\upsilon_2\omega_1)$
   iv. $1\omega_1 = \omega_1$
2) The right module over Υ (or it can be written right Υ-module) is an abelian group Ω with a scalar multiplication mapping
$$\circ : \Omega \times \Upsilon \to \Omega$$
that satisfies the following axioms (it will be written $\circ(\omega_1, \upsilon) \mapsto \omega_1\upsilon$ for scalar multiplication, where υ is an element of Υ and ω_1 is an element of Ω). Just like the left Υ-modul, again υ_1, υ_2 are arbitrary elements of Υ and ω_1, ω_2 are arbitrary members of Ω.
   i. $(\omega_1 + \omega_2)\upsilon = \omega_1\upsilon + \omega_2\upsilon$
   ii. $\omega_1(\upsilon_1 + \upsilon_2) = \omega_1\upsilon_1 + \omega_1\upsilon_2$
   iii. $\omega_1(\upsilon_1\upsilon_2) = (\omega_1\upsilon_1)\upsilon_2$
   iv. $\omega_1 1 = \omega_1$

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The focus of this study will discuss the application of free modules so that further attention will be directed to introduction of the concept of free modules. In short, a free module is a module that has a basis. This is where the understanding of vector spaces must be strong because it will be carried over to its application. In vector spaces, we know basis and the assumption is that you already know about the concept of basis in vector spaces. Then, we can go into the concept of the basis in modules. Before that, here is the formal definition of linearly dependence in a module.

Definition 4 [4] Linearly Dependence in a Module
Suppose that Υ is a ring and Ω is an Υ-module. A subset $\Psi \subseteq \Omega$ is said to be Υ-linearly dependent if there are distinct $\psi_1, \psi_2, \dots, \psi_n \in \Psi$ and $\upsilon_1, \upsilon_2, \dots, \upsilon_n \in \Upsilon$, not all of which are 0, then

$$\upsilon_1\psi_1 + \upsilon_2\psi_2 + \cdots + \upsilon_n\psi_n = 0 \tag{1}$$

A set that is not Υ-linearly dependent is said to be Υ-linearly independent.

After that, this is the definition of basis in modules. Here is the generalization of the concept of a basis in vector spaces to modules over rings.

Definition 4 [4] Basis of a Module over a Ring
Suppose an Υ-module Ω and the set Ψ which is a subset of Ω. Ψ is said to be a basis of Ω if Ψ generates Ω as an Υ-module and if Ψ is Υ-linearly independent. Therefore, $\Psi \subseteq \Omega$ is a basis of Ω if and only if $\Omega = \{0\}$, in which case $\Psi = \emptyset$ is a basis, or $\Omega \neq \{0\}$ and

i. For all $\psi \in \Omega$ can be written as

$$\psi = \upsilon_1\psi_1 + \upsilon_2\psi_2 + \cdots + \upsilon_n\psi_n \tag{2}$$

for $\psi_1, \psi_2, \dots, \psi_n \in \Psi$ and $\upsilon_1, \upsilon_2, \dots, \upsilon_n \in \Upsilon$, and

ii. Whenever there is an equation

$$\upsilon_1\psi_1 + \upsilon_2\psi_2 + \cdots + \upsilon_n\psi_n = 0 \tag{3}$$

in which $\psi_1, \psi_2, \dots, \psi_n$ are distinct elements of Ψ and $\upsilon_1, \upsilon_2, \dots, \upsilon_n \in \Upsilon$, then

$$\upsilon_1 = \upsilon_2 = \cdots = \upsilon_n = 0 \tag{4}$$

Definition 5 [4] Free Modules
An Υ-module Ω is called as a free Υ-module if it has a basis.

Here is a simple example of a free module.

Example [4]
The free module R^n has a basis $S = \{e_1, \dots, e_n\}$, in which

$$e_i = (0, \dots, 0, 1, 0, \dots, 0) \tag{5}$$

with a 1 in the i^{th} position.

After understanding the concept of free modules, i.e. when the module has a basis then recall that in vector spaces the basis plays a very important role in the analysis of algebraic structures. This means that free modules also have an essential role in the analysis of algebraic structures, so let's look at the application of free modules in some of the fields discussed in this research further.

II. Methodology

In general, this article was written using the Systematic Literature Review method. This method relies on literature studies because its main sources are journals or previous research on the same topic [5]. This study focuses on free modules and their applications. Free modules, which have basis, play a fundamental role in the analyzing algebraic structures.

Literature searches were conducted through several academic databases, such as Scopus, ScienceDirect, SpringerLink, IEEE Xplore, arXiv, and Google Scholar. Several textbook references were also used to provide a foundation for conceptual understanding and further study, including those

published by John Wiley & Sons, Springer-Verlag, and CRC Press. These books provide basic studies, particularly in the field of algebra.

The criteria used in selecting the literature included literature that directly or indirectly discusses the application of free modules in applied mathematics. Other criteria included literature written in English, reputable scientific journals, and indexed proceedings. Keywords used included *free modules* and *applications in applied fields*, including system control, cryptography, and mathematical physics. The references used for the application of free modules in applied mathematics ranged from 2007 to 2025.

This analysis is used in broader applied science. Therefore, researchers searched for previous literature related to examples of the application of free modules, either explicitly or implicitly mentioned. The researcher will then provide a brief discussion of these examples to illustrate what the application is like, which will be useful for future researchers who need a clearer picture.

III. The Application of Free Modules

Based on the results of the literature review, which involved collecting previous literature examining the application of free modules either directly or implicitly, several fields that frequently use free modules in their discussions were obtained. These fields include applications in system control theory, representation theory, cryptography and mathematical physics.

A. The Application of Free Modules in System Control Theory

Before discussing the application of free modules in system control theory, we will provide a brief and simple introduction to system control. A system is made up of different parts that work together to achieve a specific goal, it doesn't have to be physical [6]. The control system comprises subsystems and processes that incorporate the desired subsystem and process units, delivering the expected performance, based on predetermined inputs [7].

Then, what is the correlation to free modules? Free modules are algebraic objects that are modules with a basis. Although the application of free modules in system control is not a major topic, free modules implicitly provide a useful working structure in nonlinear systems and stability analysis. We will see the role of the structure then. In system control theory, the free module structure is applied to nonlinear systems through the use of free module basis, such as $\hat{Z}(x)$, which is a free module basis over the differential ring P, it is also used in free modules over the differential ring $\mathcal{D}\left[\frac{d}{dt}\right]$ [8] [9]. The vector $Z(x)$ is a monomial vector of a larger size, while $\hat{Z}(x)$ is a smaller version with a lower size and degree. This makes it easier to design the control and analyze stability using the flatness method [8].

In addition, the concept of free modules is implicitly applied on the assumption. It is the matrix containing the results of evaluating the monomial vectors from the experimental data has full row rank. This indicates that the monomial vectors generated from the data form a linearly independent set, similar to the basis elements in free modules, which also fulfil the linear independence property [10].

Unfortunately, journals discussing the application of free modules in control theory, especially in relation to flat systems, are very rare. Even within the last ten years, compared to discussions of system control theory in general. Specific discussions can be found in previous journals. The structure of free modules plays a crucial role in system control theory, particularly in the characterisation of flat systems.

A system is said to be flat if the module representing it is a free module [11]. Flat systems have a variety of key applications in the field of system control theory, including motion planning, tracking, and optimal control [11]. The concept of free modules appears implicitly in different literatures on nonlinear control theory approaches, such as flatness-based control and differential algebra [12].

B. The Application of Free Modules in Representation Theory

As the title suggests, it is important to first understand the concept of representation theory, which is closely related to algebra, before delving into the discussion of the application of free modules in a deeper theory of representation. In the 20th century, representation theory emerged as a topic discussion: in the 19th, groups were generally understood to be subsets of permutations of a set or automorphisms

$GL(V)$ of a vector space V, which are closed under composition and inverse [13]. In the definition, it is very clearly stated as "vector space", so let's look at the application of free modules in representation theory from several literature sources.

In representation theory, free modules are applied to describe algebraic actions and structure the desired representation [14]. This can be seen in literature that indirectly applies the concept of free modules to construct multiplicity-free weight modules over quantum affine algebras [14]. Another reference uses free modules applied in representation theory to explore preprojective algebra [15] [16].

Using free local modules helps to categorise the different types of module and understand how they interact with each other [15]. The freedom of a module denotes its rank varieties, if the module is projective or free in algebraic terms, then its rank variety is $\{0\}$ [17]. Conversely, the absence of the module is evident in the non-trivial rank variety [17]. In various literature, the free module structure is used to provide an ordered representation, an ordered representation of a finite group is defined as the group's action on a free vector space of dimension equal to the group's element count, with the group elements forming basis [18].

C. The Application of Free Modules in Cryptography

This section will introduce you to the concept of cryptography and the application of free modules in it. Cryptography is a field of science that aims to hide the meaning of a message [19]. "The Concise Oxford English Dictionary defines cryptography as "the art of writing or solving codes" as cited in [20].

In contemporary language, codes are known as methods of securing information, which are commonly referred to as "encryption" [20]. Cryptography is closely related to algebra, including free modules. So, let's take a look at how free modules are used in cryptography. In cryptography, particularly in the context of lattice-based cryptography, free modules are employed to describe a lattice as a free module over an order in divisor algebra, this approach enables the generalisation of various classical concepts in lattice reduction theory [21].

In elliptic curve-based cryptography, the set of points on the curve constitutes an Abelian group [22]. Conceptually, this can be viewed as a free module over $\mathbb{Z}$ forming the basis for cryptographic operations, particularly those involving the dot sum in cryptographic algorithms [23]. In cryptography, particularly in coding theory, free modules are used to create various types of code for cryptographic systems [24].

D. The Application of Free Modules in Mathematical Physics

Mathematical physics has long formed the basis of significant advances in physics and mathematics, and continues to drive progress in these fields [25]. One of the three main topics covered in Modern Mathematical Physics is mathematical methods in physics [25]. Let's take a closer look at the application of free modules in mathematical physics.

The fusion ring R is a free module structure over $\mathbb{Z}$ with a finite basis and modular data [26]. It plays a vital role in mathematical physics and tensor categories, which in modular tensor categories with a ribbon structure are crucial for applying basic category theory to mathematical physics [26]. In other studies, free modules are also indirectly present as vector bundles over flag manifolds that have a fibre vector space which in the analysis is assumed to be $\mathbb{R}^n$ or $\mathbb{C}^n$ [27] [28].

Another literature exists that studies resolution or flag structures, where the literature utilizes the structure of differential modules. The structure of differential modules which are often constructed as free modules or free differential modules in this context [29]. This theoretical structure is important for forming minimal free resolutions and performing homology analyses [29].

Table 1. Summary of the Application of Free Modules in the Applied Mathematical Science

Fields	Example Use	Key References
System Control Theory	Free module basis on nonlinear systems for control analysis.	Data-Driven stabilization of nonlinear polynomial systems with noisy data, 2022.

Fields	Example Use	Key References
		Flatness-based control revisited: The *HEOL* setting, 2024.
	The concept of independent modules is implemented through the assumption that matrices with full row rank represent a basis of linearly independent monomial vectors.	Learning control for polynomial systems using sum of squares relaxations, 2020.
	Flat systems are represented by free modules and are used for optimal trajectory planning, tracking, and control.	Computation of bases of free modules over the Weyl algebras, 2007.
	Free modules appear in flatness-based and differential algebra-based control approaches.	Model-Free Control and Intelligent PID Controllers: Towards A Possible Trivialization of Nonlinear Control, 2009.
Representation Theory	Free modules are used to represent algebraic actions.	On Multiplicity-Free Weight Modules Over Quantum Affine Algebras, 2024.
	Free modules are applied to explore preprojective algebra.	On the Module Categories of Generalized Preprojective Algebras of Dynkin type, 2022.
		Dimensions of τ-Tilting Modules Over Path Algebras and Preprojective Algebras of Dynkin Type, 2025.
	The free module is used to determine the rank variety with a value of {0}.	Small Modules with Interesting Rank Varieties, 2023.
	Free modules are used to construct ordered representations of finite groups.	Forms of Nice Questions, 2025.
Cryptography	Free modules are applied to represent lattices and generalize classical lattice theory.	Reduction Theory of Algebraic Modules and their Successive Minima, 2021.
	Points on an elliptic curve are viewed as free modules over $\mathbb{Z}$ on which cryptographic operations are based.	An Elliptic Curve Cryptography based Algebraic Structures for Security Provisioning using Algebraic Cryptography, 2025.
	Free modules are applied to form codes in cryptography.	Application of Module to Coding Theory: A Systematic Literature Review, 2024.
		Density of Free Modules Over Finite Chain Rings, 2022.
Mathematical Physics	The fusion ring as a free module over $\mathbb{Z}$ is used in mathematical physics of the tensor category.	Classification of Integral Modular Data Up to Rank 13, 2024.

Fields	Example Use	Key References
	Free modules exist as vector bundles over the flag manifold.	Gauge Theories and Fiber Bundles: Definitions, Pictures, and Results, 2022. Ricci-Flat Metrics on Vector Bundles Over Flag Manifolds, 2020.
	The free module is used to form minimal free resolution and homology analysis.	Minimal Free Resolutions of Differential Modules, 2022.

The Table 1 shows that free modules, which are modules with a basis, play a role in various fields. The commonality is that the basis of the free module itself is often found in the fields studied. The concept of a free module, implicitly present, is also found in some literature.

IV. Conclusion

A module is an extension of a vector space. If a module has a basis, it is said to be free. Free modules play a role not only in abstract algebra theory, but also in other branches of mathematics. Understanding the concept and structure of free modules is essential for developing applied mathematics based on algebra. Free modules are frequently applied in various fields such as system control theory, representation theory, cryptography and mathematical physics. In system control theory, free modules, particularly their structures, are used to model flat systems, which are important in trajectory planning and optimal control. In representation theory, free modules facilitate the classification and construction of more complex modules.

In cryptography, the structure of free modules forms the basis of lattice representation and point operations on elliptic curves. In the field of mathematical physics, meanwhile, free modules are present in algebraic structures such as fusion rings, as well as in the construction of minimal free resolutions. Therefore, understanding the concept and structure of free modules can aid the development of applied mathematics, which essentially uses algebra as a theoretical foundation.

From this literature review, free modules are widely applied in system control theory, and are also frequently found in representation theory and cryptography. However, the application of free modules in mathematical physics remains underexplored. Therefore, future researchers can examine the applications of free modules, such as integration with computational algebra, quantum computing, or AI-based modeling, to expand their application in applied mathematics. For applied mathematicians, this review can have practical implications, as understanding the structure of free modules can strengthen the modeling and analysis of complex systems.

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