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A Study on Strategic Approaches to Team NIM Games Using Number Theory

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Abstract

Many games are related to Maths, such as *bridge* card games, dominoes, chess, NIM, and many others. For example, the NIM game is a strategy game played by two players or two teams in which each player or team takes turns taking at least one object with certain rules. Winning the game depends on how many objects are available and who plays first. In this article, we will present the NIM Maxima game. In this NIM game, the concept of Mathematics that will be used is number theory to determine a strategy to win the game.

Keywords: NIM Games; Strategy; Congruent

I. Introduction

A game is a recreational activity that is usually played by two or more people. NIM games are classic games that rely on strategy as their main element [1]. NIM game originated in China, the name is derived from the word 'NIMM' which in German means 'to take' [2] [3] [4] [5] [6] [7] [8] [9]. NIM is usually played as a misere game where the player who takes the last object loses. NIM can also be played normally, where the player who picks up the last object wins. NIM is a type of game that has existed for centuries around the world and Europeans first referenced the game in the 15th century. NIM is a game played by two people with certain rules. In this article, we will discuss a strategy to win a maximal NIM game where the maximal NIM game is slightly modified into a team game with each team consisting of 2 or more players. In this article, a team NIM Maxima game will be presented.

In the NIM Maksima team game, a strategy will be determined to win the game as a team. In determining the strategy in this team NIM Maksima game, some mathematical concepts that appear in this game will be used, the mathematical concepts used are about number theory. The concept of number theory, especially on congruence will be used in forming the winning strategy of NIM Maxima team. The description in this paper can be seen that the concept of number theory can be applied in the NIM team game. So, the purpose of this paper is to apply the concepts of number theory in determining a winning strategy in the NIM Maksima team game, before the NIM game starts, the conditions and rules that have been set beforehand.

II. Theoretical Study

This chapter will discuss what a game is, NIM games, maximal NIM games, and Number Theory as follows:

A. Games

Games are activities carried out by 2 or more people. According to references [1] [2] [4] [5] [6] [7] [10], something can be said to be a game if it has the following characteristics:

a. Have at least 2 players
b. Opportunity to take turns playing

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c. One player's loss is another player's gain
d. The player knows all the information of the game
e. Does not contain the element of chance as in the dice game.

B. History of NIM Game

This game has been around since ancient times. The game of Nim originated in China called Tsyanshidzi or picking up stones. The name comes from the word Nim which in German means take or grab [2] [4] [5] [7] [8] [11]. Nim is a type of game that has existed for centuries around the world and Europeans first referenced the game in the 15th century. It was a Harvard mathematics professor, Charles Bouton, who developed the Nim game theory.

C. Number Theory

Number theory is a branch of pure mathematics that deepens the understanding of integers or integer-valued functions [5] [12] [13] [14] [15] [16] [17] [18] [19] [20].

D. Number

Various types of numbers are studied in Maths. These types of numbers include:

1. Natural numbers ($\mathbb{N}$), i.e. numbers that start from 1 and next $\{1, 2, 3, 4, 5, 6, 7, \dots\}$. Based on their form, natural numbers can be further divided into four types, namely:
   a. Even numbers $\{2, 4, 6, 8, 10, \dots\}$.
   b. Odd numbers $\{1, 3, 5, 7, 9, \dots\}$.
   c. A prime number is a natural number that has only two factors, one and itself $\{2, 3, 5, 7, 11, \dots\}$.
   d. A composite number that has more than two factors or is divisible by a number other than one and itself $\{4, 6, 8, 9, 10, \dots\}$.
2. Integers numbers ($\mathbb{Z}$), which is an natural numbers added with a zero and negative number
   $$\mathbb{Z} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}.$$
3. Fractional numbers, which are numbers that can be expressed in fraction form
   $$a/b = \{1/2, 3/4, 5/6, 7/8, \dots\}.$$
4. Rational numbers ($\mathbb{Q}$), which include Integers numbers and Fractions numbers.
5. Irrational numbers, which are numbers that cannot be expressed in fraction form
   $$a/b = \{\sqrt{2}, \sqrt{3}, dan\ bilangan\ irasional\ lainnya\}$$
6. Real numbers ($\mathbb{R}$), is a number that includes rational and irrational numbers.
7. Imaginary Numbers, which are numbers that include i with $i^2 = -1$,
   Example 2.1 $\sqrt{-1}, \sqrt{-2}$.
8. Complex Numbers ($\mathbb{C}$), is a number that includes Real and Imaginary numbers.

III. Methods

In this chapter, we will discuss some properties of integers that are used as methods for the discussion of this article. Integers are numbers that do not have decimal fractions, for example $8, 21, 8765, -34, 0$. The inverse of an integers are real numbers that have decimal points, for example $8.0, 34.25, 0.02$.

A. Division Properties of Integers

Given a and b, with $b \neq 0$. It is said a divides b if exist integer number c such that $b = ac$. Denoted by $a \mid b$.

Example 2.2 3 divides 21 is exist integer number $c = 7$ such that $3 \mid 21$, because $21/3 = 7$ (integer number) or $21 = 3 \times 7$.

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Theorem 2.3 [12] [13] [20] *Given integers a and b, with $b > 0$, there exist unique integers q and r satisfying*

$$a = qb + r \qquad 0 \leq r < b \tag{1}$$

The integers q and r are called, respectively, the quotient and remainder in the division of a by b.
Example 2.4 1876 divided by 47 exist $q = 39$ and $r = 43$, such that $1876 = 39 .47 + 43$

B. Greatest Common Divisor (gcd)

Definition 2.5 [12] [13] [20] *Let a and b be given integers, with at least one of them different from zero. The greatest common divisor of a and b, denoted by* $\gcd(a, b)$*, is the positive integer d satisfying the following:*

(a) $d|a$ and $d|b$.
(b) If $c|a$ and $c|b$, thenm $c \leq d$.

Example 2.6 $\gcd(54,63) = 9$ because 9 is the largest of these integers can divides 9 | 54 and 9|63.

Theorem 2.7 [12] [13] [20] *Given integers a and b, not both of which are zero, there exist inttegers x and y such that*

$$\gcd(a, b) = ax + by \tag{2}$$

C. Relatively Prime

Definition 2.8 [12] [13] [20] *two integers a and b, not both of which zero, are said to be relatively prime whenever* $\gcd(a, b) = 1$

Example 2.9 23 and 15 relatively prime, because $\gcd(23,15) = 1$. However, 20 and 15 not relatively prime because $\gcd(20,15) = 5 \neq 1$.

Theorem 2.10 [12] [13] [20] *Let a and b be integers, not both zero. Then a and b are relatively prime if and only if there exist integers x and y such that $1 = xa + yb$.*

Example 2.11 Number 15 and 7 relatively prime, because $\gcd(15,7) = 1$, or can write $1.15 + (-2).7 = 1$ which $x = 1$ and $y = -2$. But, 15 and 5 no relatively prime $\gcd(15,5) = 5 \neq 1$ such that 1 and 5 can't denoted by $x .20 + y .5 = 1$.

D. congruent

Definition 2.12 [12] [13] [20] *let m be a fixed positive integer. Two integers a and b are said congruent modulo m, symbolized by*

$$a \equiv b \ (mod\ m) \tag{3}$$

if m divides the difference $a - b$; that is, provided that $a - b = km$ for some integer k.

Theorem 2.13 [12] [13] [20] *let $m > 1$ be a fixed and a, b, c, d be arbitrary integers. Then the following properties hold:*

1. $a \equiv a \ (mod\ m)$
2. *If* $a \equiv b \ (mod\ m)$*, then* $b \equiv a \ (mod\ m)$.
3. *If* $a \equiv b \ (mod\ m)$ *and* $b \equiv c \ (mod\ m)$*, then* $a \equiv c \ (mod\ m)$.
4. *If* $a \equiv b \ (mod\ m)$ *and* $c \equiv d \ (mod\ m)$*, then* $a + c \equiv b + d \ (mod\ m)$ *and* $ac \equiv bd \ (mod\ m)$.
5. *If* $a \equiv b \ (mod\ m)$*, then* $a + c \equiv b + c \ (mod\ m)$ *and* $ac \equiv bc \ (mod\ m)$.
6. *If* $a \equiv b \ (mod\ m)$*, then* $a^k \equiv b^k \ (mod\ m)$ *for any positive integer k.*

Example 2.14 $23 \equiv 3 \ (mod\ 5)$, because $23 \ (mod\ 5) = 3$ and $3 \ (mod\ 5) = 3$.

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IV. Result and Discussion

This section will discuss the application of congruence theory in the NIM Maxima game. The concept of the game discussed in this study is that the player who takes the last remaining object is the winning player.

A. Team NIM Maxima Victory Strategy

In the NIM Maxima game, let's say that the player has M objects and the maximum number of times to pick up objects on each turn N, and $M > N; (M, N \in \mathbb{N})$. There are four players (call them players A_1, A_2, B_1, and B_2) with Team A and Team B, respectively. And suppose there are objects available on the table. Each player in turn takes a maximum of 3 objects. The winning strategy of NIM Maxima where the team that takes the last remaining object is the winner will be presented in two illustrations and four cases below.

Illustration of Team A as the winner:

In this problem there are two cases, namely: Suppose in the first case players from each team sit crosswise or alternately around the table. In the second case, each player sits in a team with his teammates. Suppose that each player of each team takes turns issuing $1, 2$, or 3 objects at once from a stack starting with 26 objects. The person who takes the last object wins the game. It will be shown that the first team can win the game no matter what the second team does.

Team A's first case solution:

Team A can always win the game by backward analysis. When it comes to the last step team A can win the game if team A can leave $1, 2$, or 3 objects. Team B will be directed to leave some $1, 2$, or 3 objects when it comes to removing objects from a pile containing 4 objects. As a result, before the last move, team A should be able to leave 4 objects for team B. This can be done when the remaining $5, 6$, or 7 objects before the last step, this can be obtained if team B has to remove objects from the stack by 8 objects. As a result, on the step before the last one, Team A should be able to leave 8 objects for team B. Based on the illustration, it is shown that team A can win the game if it leaves team B with a certain number of $24, 20, 16, 12, 8$, and 4 objects. This means that team A can win the game if team A leaves as many objects as $0 \ mod \ 4$ or $0 \ (mod \ n + 1)$, wich n is the maximum number of objects that can be taken by players from each team. To clarify the illustration above is presented in Table 1 below.

Table 1. Illustration NIM Maxima with 26 Objects and Team A is the Winner

Team A		**Team B**	
Objects taken	Remaining objects	Objects taken	Remaining objects
2	24	2	22
2	20	3	17
1	16	1	15
3	12	3	9
1	8	2	6
2	4	2	2
2	0		

Team A's second case solution:

Team A can always win the game by backward analysis. When it comes to the last step team A can win the game if team A can leave $1, 2, 3, 4, 5$, or 6 objects. Team B will be directed to leave some $1, 2, 3, 4, 5$, or 6 objects when it comes to removing objects from a pile containing 8 objects. As a result, before the last move, team A should be able to leave 8 objects for team B. This can be done when the remaining $10, 11, 12, 13$, or 14 objects before the last step, this can be obtained if team B has to remove objects from the stack by 16 objects. As a result, on the step before the last one, Team A should be able

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to leave 16 objects for team B. Based on the illustration, it is shown that team A can win the game if it leaves team B with a certain number of $24, 16$, and 8 objects. This means that team A can win the game if team A leaves as many objects as $0\ mod\ 8$ or $0\ (mod\ a(n+1))$, wich n is the maximum number of objects that can be taken by players from each team and a denotes the number of players from each team. To clarify the illustration above is presented in Table 2 below.

Table 2. Illustration NIM Maxima with 26 Objects and Team A is the Winner

Team A		**Team B**	
Total objects taken	Remaining objects	Total objects taken	Remaining objects
2	24	5	19
3	16	4	12
4	8	2	6
6	0		

Illustration of Team B as the winner:

In this problem there are two cases, namely: Suppose in the first case players from each team sit crosswise or alternately around the table. In the second case, each player sits in a team with his teammates. Suppose that each player of each team takes turns issuing $1, 2$, or 3 objects at once from a stack starting with 24 objects. The person who takes the last object wins the game. It will be shown that the second team can win the game no matter what the first team does.

Team B's first case solution:

Team B can always win the game by backward analysis. When it comes to the last step team B can win the game if team B can leave $1, 2$, or 3 objects. Team A will be directed to leave some $1, 2$, or 3 objects when it comes to removing objects from a pile containing 4 objects. As a result, before the last move, team B should be able to leave 4 objects for team A. This can be done when the remaining $5, 6$, or 7 objects before the last step, this can be obtained if team A has to remove objects from the stack by 8 objects. As a result, on the step before the last one, Team B should be able to leave 8 objects for team A. Based on the illustration, it is shown that team B can win the game if it leaves team A with a certain number of $20, 16, 12, 8$, and 4 objects. This means that team B can win the game if team B leaves as many objects as $0\ mod\ 4$ or $0\ (mod\ n+1\ )$, wich n is the maximum number of objects that can be taken by players from each team. To clarify the illustration above is presented in Table 3 below.

Table 3. Illustration NIM Maxima with 24 Objects and Team B is the Winner

Team A		**Team B**	
Objects taken	Remaining objects	Objects taken	Remaining objects
3	21	1	20
2	18	2	16
1	15	3	12
3	9	1	8
1	7	3	4
2	2	2	0

Team B's second case solution:

Team B can always win the game by backward analysis. When it comes to the last step team B can win the game if team B can leave $1, 2, 3, 4, 5$, or 6 objects. Team A will be directed to leave some $1, 2, 3, 4, 5$, or 6 objects when it comes to removing objects from a pile containing 8 objects. As a result, before the last move, team B should be able to leave 8 objects for team A. This can be done when the remaining $10, 11, 12, 13$, or 14 objects before the last step, this can be obtained if team A has to remove objects from the stack by 16 objects. As a result, on the step before the last one, Team B should be able to leave 16 objects for team A. Based on the illustration, it is shown that team B can win the game if it

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leaves team A with a certain number of 16 and 8 objects. This means that team A can win the game if team A leaves as many objects as 0 mod 8 or 0 $(mod\ a(n+1))$, wich n is the maximum number of objects that can be taken by players from each team and a denotes the number of players from each team. To clarify the illustration above is presented in Table 4.

Table 4. Illustration NIM Maxima with 24 Objects and Team B is the Winner

Team A		**Team B**	
Total objects taken	Remaining objects	Total objects taken	Remaining objects
2	22	6	16
3	13	5	8
4	4	4	0

V. Conclusion

After analyzing the team maxima NIM game above, it was concluded that the victory between the two teams depends on the positioning of the players from each team. The conclusions are as follows:

1. NIM Maxima wins in the case that each player from each team sits in turn, the victory of one team depends on M number of objects at the beginning of the game and N number of object picks set before the game starts:
   i. If $M \not\equiv 0\ (mod\ N+1)$, then the team that makes the first move wins the game
   ii. If $M \equiv 0\ (mod\ N+1)$, then the team that makes the second move wins the game
2. NIM Maxima wins in the case that each player sits with his teammates, the victory of one of the teams depends on M number of objects at the beginning of the game and N number of object picks set before the game starts, and a the number of players from each team:
   i. If $M \not\equiv 0\ (mod\ a(N+1))$, then the team that makes the first move wins the game
   ii. If $M \equiv 0\ (mod\ a(N+1))$, then the team that makes the second move wins the game.

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